SOS3003
Applied data analysis for social science
Lecture note 05-2010

Erling Berge
Department of sociology and political science
NTNU

Literature

- Regression criticism I
  Hamilton Ch 4 p109-123

Analyses of models are based on assumptions

- OLS is a simple technique of analysis with very good theoretical properties. But:
- The good properties are based on certain assumptions
- If the assumptions do not hold the good properties evaporates
- Investigating the degree to which the assumptions hold is the most important part of a regression analysis

OLS-REGRESSION: assumptions

- I SPECIFICATION REQUIREMENT
  - The model is correctly specified
- II GAUSS-MARKOV REQUIREMENTS
  - Ensures that the estimates are "BLUE"
- III NORMALLY DISTRIBUTED ERROR TERM
  - Ensures that the tests are valid

I SPECIFICATION REQUIREMENT

- The model is correctly specified if
  - The expected value of y, given the values of the independent variables, is a linear function of the parameters of the x-variables
  - All included x-variables have an impact on the expected y-value
  - No other variable has an impact on expected y-value ***at the same time as they correlate with included x-variables***

II GAUSS-MARKOV REQUIREMENTS (i)

(1) x is known, without stochastic variation
(2) Errors have an expected value of 0 for all i

- $E(\varepsilon_i)=0$ for all i

Given (1) and (2) ε_i will be independent of x_k for all k and OLS provides **unbiased estimates** of β
(unbiased = forventningsrett)

II GAUSS-MARKOV REQUIREMENTS (ii)

(3) Errors have a constant variance for all i

- $Var(\varepsilon_i) = \sigma^2$ for all i

This is called homoscedasticity

(4) Errors are uncorrelated with each other

- $Cov(\varepsilon_i, \varepsilon_j)=0$ for all i ≠ j

This is called no autocorrelation

II GAUSS-MARKOV REQUIREMENTS (iii)

Given (3) and (4) in addition to (1) and (2) provides:

- a. Estimates of standard errors of regression coefficients are unbiased and
- b. The **Gauss-Markov theorem**:

OLS estimates have **less variance** than any other linear unbiased estimate (including ML estimates)

OLS gives "BLUE"

(**B**est **L**inear **U**nbiased **E**stimate)

II GAUSS-MARKOV REQUIREMENTS (iv)

(1) - (4) are called the GAUSS-MARKOV requirements

- Given (2) - (4) with an additional requirement that errors are uncorrelated with x-variables:
  - $\text{cov}(x_{ik}, \varepsilon_i) = 0$ for all i,k

The coefficients and standard errors are consistent (converging in probability to the true population value as sample size increases)

Footnote 1:
Unbiased estimators

- Unbiased means that

  $E[b_k] = \beta_k$
- In the long run we are bound to find the population value - β_k - if we draw sufficiently many samples, calculates b_k and average these

Footnote 2:
Consistent estimators

- An estimator is consistent if we as sample size (n) grows towards infinity, find that b approaches β and s_b [or SE_b] approaches σ_β
- b_k is a consistent estimator of β_k if we for any small value of c have

  $\lim_{n\to\infty} [\Pr\{ |b_k - \beta_k| < c \}] = 1$

Footnote 3: In BLUE ”Best” means minimal variance estimator

- Minimal variance or efficient estimator means that

  $var(b_k) < var(a_k)$ for all estimators a different from b
- Equivalent:

  $E[b_k - \beta_k]^2 < E[a_k - \beta_k]^2$ for all estimators a unlike b

Footnote 4: Biased estimators

- Even if the requirements ensuring that our estimates are BLUE one may at times find biased estimators with less variance such as in
- Ridge Regression

Footnote 5: Non-linear estimators

- There may be **non-linear estimators** that are unbiased and with less variance than BLUE estimators

III NORMALLY DISTRIBUTED ERROR TERM

- (5) If all errors are normally distributed with expectation 0 and standard deviation of σ^2 , that is if

  $\varepsilon_i \sim N(0, \sigma^2)$ for all i

  - Then we can test hypotheses about β and σ, and
  - OLS estimates will have less variance than estimates from all other unbiased estimators
  - **OLS results are "BUE"**

(Best Unbiased Estimate)

Problems in regression analysis that cannot be tested

- If all relevant variables are included
- If x-variables have measurement errors
- If the expected value of the error is 0

  This means that we are unable to check if the correlation between the error term and x-variables actually is 0

  OLS constructs residuals so that $cov(x_{ik},e_i)=0$

  This is in reality saying the same as the first point that we are unable to test if all relevant variables are included

Problems in regression analysis that can be tested (1)

- Non-linear relationships
- Inclusion of an irrelevant variable
- Non-constant variance of the error term (heteroscedasticity)
- Autocorrelation for the error term
- Correlations among error terms
- Non-normal error terms
- Multicollinearity

Consequences of problems (Hamilton, p113)

		Unwanted properties of estimates			
Require ment	**Problem**	Biased estimate of b	Biased estimate of SE_b	Invalid t&F-tests	High var[b]
Specification	**Non-linear relationship**	X	X	X	-
-"-	**Excluded relevant variable**	X	X	X	-
-"-	**Included irrelevant variable**	0	0	0	X
Gauss-Markov	**X with measurement error**	X	X	X	-
-"-	**Heteroscedasticity**	0	X	X	X
-"-	**Autocorrelation**	0	X	X	X
-"-	**X correlated with ε**	X	X	X	-
Normal distribution	**ε not normally distributed**	0	0	X	X
... no requirement	**Multicollinearity**	0	0	0	X

Problems in regression analysis that can be discovered (2)

- Outliers (extreme y-values)
- Influence (cases with large influence: unusual combinations of y and x-values)
- Leverage (potential for influence)

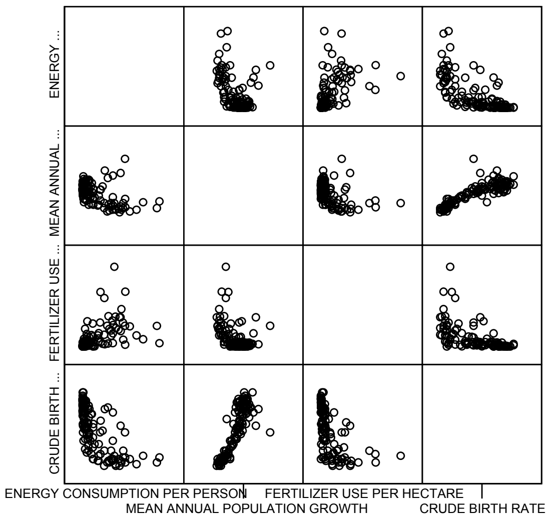
Tools for discovering problems

- Studies of
  - One-variable distributions (frequency distributions and histogram)
  - Two-variable co-variation (correlation and scatter plot)
  - Residual (distribution and covariation with predicted values)

Correlation and scatter plot

Data from 122 countries		ENERGY CONSUMPTION PER PERSON	MEAN ANNUAL POPULATION GROWTH	FERTILIZER USE PER HECTARE	CRUDE BIRTH RATE
ENERGY CONSUMPTION PER PERSON	Pearson Correlation	1	-,505	,533	-,689
	N	125	122	125	122
MEAN ANNUAL POPULATION GROWTH	Pearson Correlation	**-,505**	1	-,469	,829
	N	122	125	125	125
FERTILIZER USE PER HECTARE	Pearson Correlation	**,533**	**-,469**	1	-,589
	N	125	125	128	125
CRUDE BIRTH RATE	Pearson Correlation	**-,689**	**,829**	**-,589**	1
	N	122	125	125	125

Correlation and scatter plot



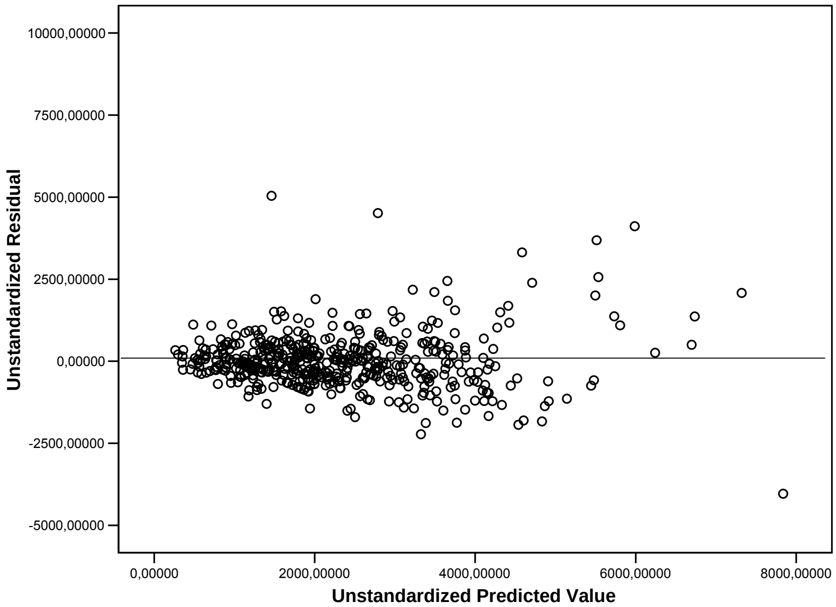
Heteroscedasticity

(non-constant variance of error term) can arise from:

- Measurement error (e.g. y more accurate the larger x is)
- Outliers
- If ε_i contains an important variable that varies with both x and y (specification error)
- Specification error is the same as the wrong model and may cause heteroscedasticity
- An important diagnostic tool is a plot of the residual against predicted value ($\hat{Y}$)

Example: Hamilton table 3.2

Dependent Variable: Summer 1981 Water Use	Unstandardized Coefficients			
	B	Std. Error	t	Sig.
(Constant)	242,220	206,864	1,171	,242
Income in Thousands	20,967	3,464	6,053	,000
Summer 1980 Water Use	,492	,026	18,671	,000
Education in Years	-41,866	13,220	-3,167	,002
head of house retired?	189,184	95,021	1,991	,047
# of People Resident 1981	248,197	28,725	8,641	,000
Increase in # of People	96,454	80,519	1,198	,232



From the regression reported in table 3.2 in Hamilton

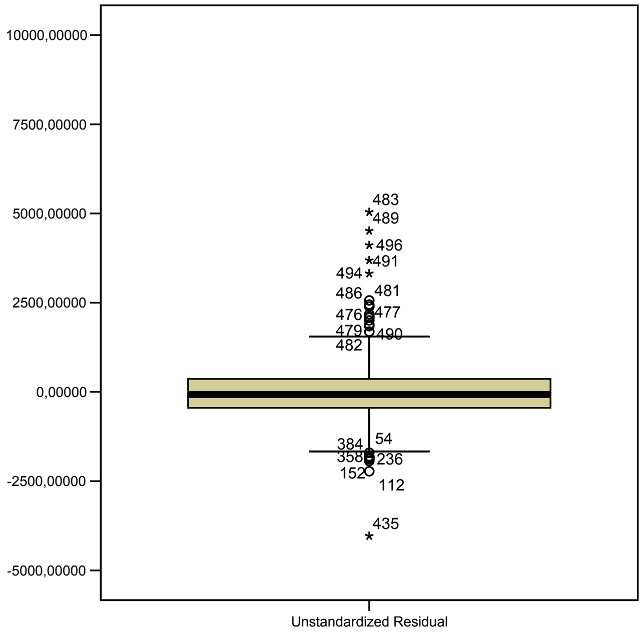
Footnote for the previous figure

- There is heteroscedasticity if the variation of the residual (variation around a typical value) varies systematically with the value of one or more x-variables
- The figure shows that the variation of the residual increases with increasing predicted y: $\hat{y}$
- Predicted y ($\hat{y}$) is in this case an index showing high average x-values
- When the variation of the residual varies systematically with the values of the x-variables like this, we conclude with heteroscedasticity

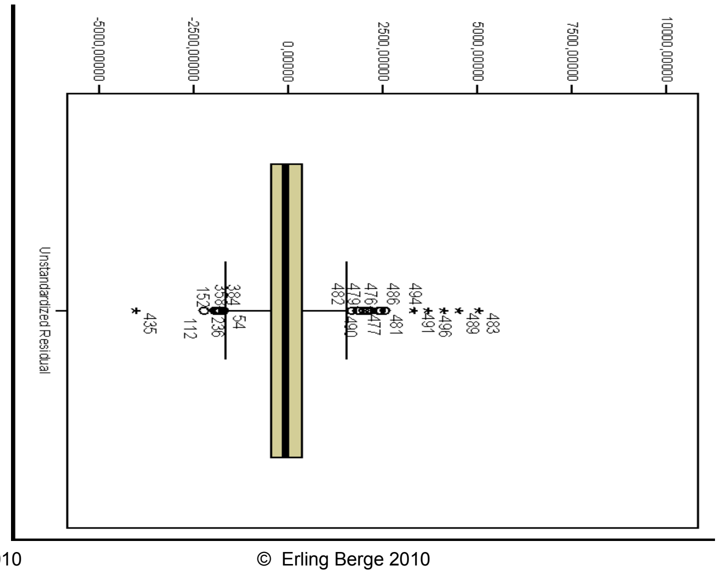
Box-plot of the residual shows

- Heavy tails
- Many outliers
- Weakly positively skewed distribution

Will any of the outliers affect the regression?

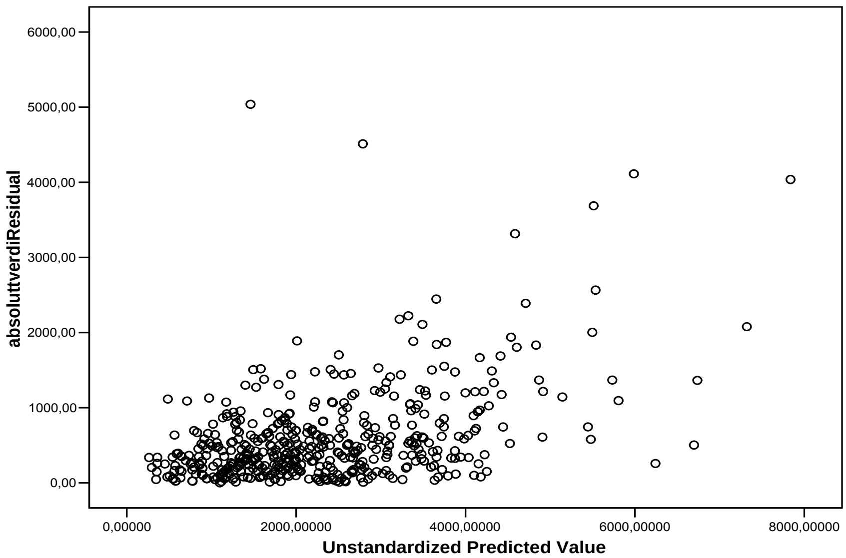


The distribution seen from another angle



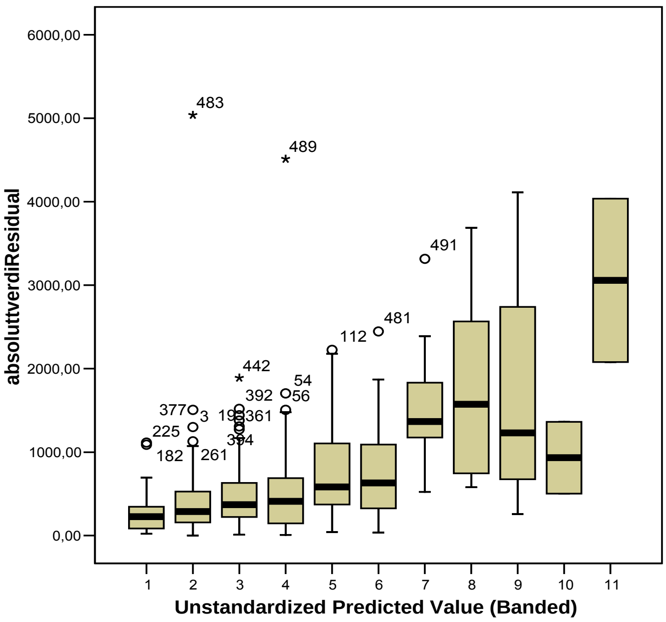
Band-regression

- Homoscedasticity means that the median (and the average) of the absolute value of the residual, i.e.: median{$|e_i|$}, should be about the same for all values of the predicted y_i
- If we find that the median of $|e_i|$ for given predicted values of y_i changes systematically with the value of predicted y_i ($\hat{y}_i$) it indicates heteroscedasticity
- Such analyses can easily be done in SPSS



Absolute value of e_i (Based on regression in table 3.2 in Hamilton)

Approximate band regression (cpr figure 4.4 in Hamilton)



Band regression in SPSS

- Start by saving the residual and predicted y from the regression
- Compute a new variable by taking the absolute value of the residual (Use “compute” under the “transform” menu)
- Then partition the predicted y into bands by using the procedure ”Visual bander” under the ”Transform” menu
- Then use ”Box plot” under ”Graphs” where the absolute value of the residual is specified as variable and the band variable as category axis

Footnote to Eikemo and Clausen 2007

- Page 121 describes White's test of Heteroscedasticity
- The description **is wrong**
- They say to replace y with e^2 in the regression on all the x variables
- That is not sufficient.
- The x-variables have to be replaced by all unique cross products of x with x (including x^2)
- Unique elements of the Kronecker product of x with x (where x is the vector of x-variables)

Autocorrelation (1)

- Correlation among variable values on the same variable across different cases

  (e.g. between ε_i and ε_{i-1})
- Autocorrelation leads to larger variance and biased estimates of the standard error - similar to heteroscedasticity
- In a simple random sample from a population autocorrelation is improbable

Autocorrelation (2)

- Autocorrelation is the result of a wrongly specified model. A variable is missing
- Typically it is found in time series and geographically ordered cases
- Tests (e.g. Durbin-Watson) is based on the sorting of the cases. Hence:
- A hypothesis about autocorrelation needs to specify the sorting order of the cases

Durbin-Watson test (1)

$$d = \frac{\sum_{i=2}^{n} \left(e_i - e_{i-1} \right)^2}{\sum_{i=1}^{n} e_i^2}$$

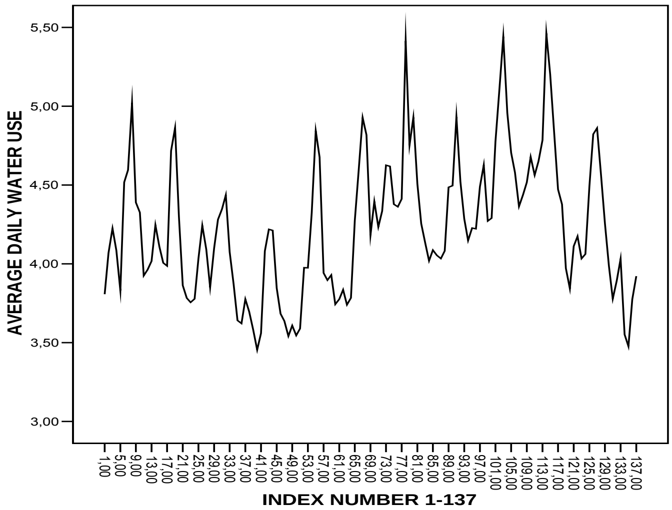
Should not be used for **autoregressive models**, i.e. models where the y-variable also is an x-variable, see table 3.2

Durbin-Watson test (2)

- The sampling distribution of the d-statistic is known and tabled as **d_L** and **d_U** (table A4.4 in Hamilton), the number of degrees of freedom is based on n and K-1
- Test rule:
  - Reject if $d<d_L$
  - Do not reject if $d>d_U$
  - If $d_L < d < d_U$ the test is inconclusive
- d=2 means uncorrelated residuals
- Positive autocorrelation results in d<2
- Negative autocorrelation results in d>2

Daily water use, average pr month

Example:



Ordinary OLS-regression where the case is month

Dependent Variable: AVERAGE DAILY WATER USE	Unstandardized Coefficients		t	Sig.
	B	Std. Error		
(Constant)	3,828	,101	38,035	,000
AVERAGE MONTHLY TEMPERATURE	,013	,002	7,574	,000
PRECIPITATION IN INCHES	-,047	,021	-2,234	,027
CONSERVATION CAMPAIGN DUMMY	-,247	,113	-2,176	,031

Predictors: (Constant), CONSERVATION CAMPAIGN DUMMY, AVERAGE MONTHLY TEMPERATURE, PRECIPITATION IN INCHES

Test of autocorrelation

Dependent Variable: AVERAGE DAILY WATER USE	R	R Square	Adjusted R Square	Std. Error of the Estimate	Durbin-Watson
1	,572(a)	,327	,312	,36045	**,535**

Predictors: (Constant), CONSERVATION CAMPAIGN DUMMY, AVERAGE MONTHLY TEMPERATURE, PRECIPITATION IN INCHES

N = 137, K-1 = 3

Find limits for rejection / acceptance of the null hypothesis of no autocorrelation with level of significance 0,05

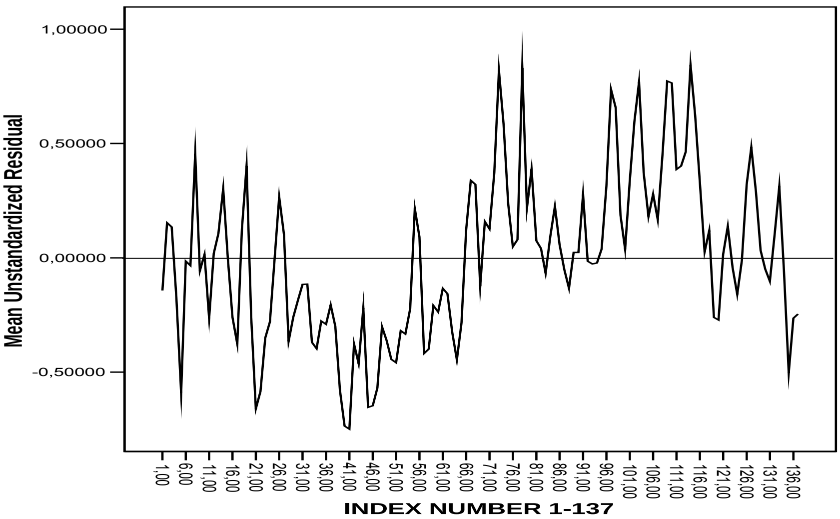
Tip: Look up table A4.4 in Hamilton, p355

Autocorrelation coefficient

m-th order autocorrelation coefficient

$$r_m = \frac{\sum_{t=1}^{T-m}\left(e_t - \bar{e}\right)\left(e_{t+m} - \bar{e}\right)}{\sum_{t=1}^{T}\left(e_t - \bar{e}\right)^2}$$

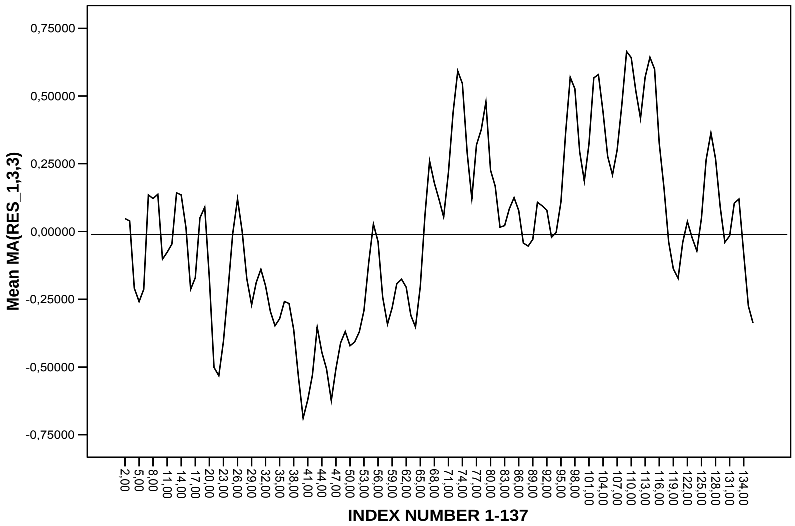
Residual ”Daily water use”, month



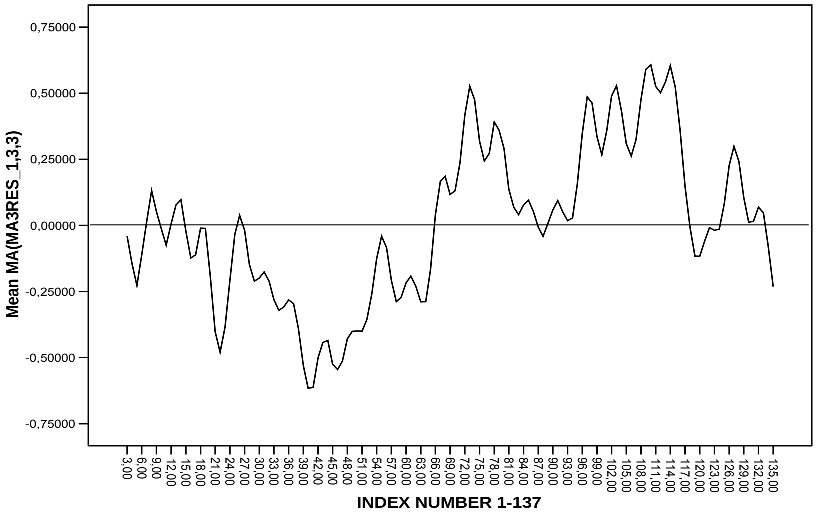
Smoothing with 3 points

- Sliding average $$e_t^* = \frac{e_{t-1} + e_t + e_{t+1}}{3}$$
- "Hanning" $$e_t^* = \frac{e_{t-1}}{4} + \frac{e_t}{2} + \frac{e_{t+1}}{4}$$
- Sliding median $$e_t^* = median\{e_{t-1}, e_t, e_{t+1}\}$$

Residual, smoothing once

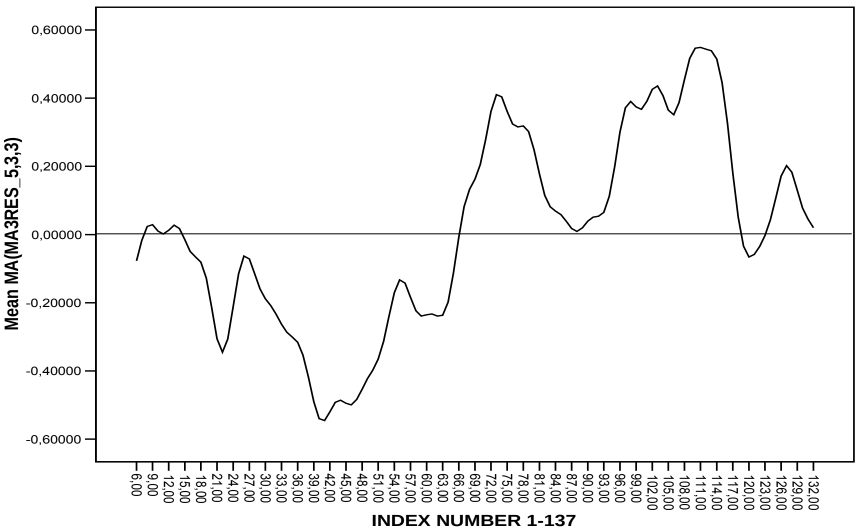


Residual, smoothing twice



Spring 2010 © Erling Berge 2010 45

Residual, smoothing five times



Spring 2010 © Erling Berge 2010 46

Consequences of autocorrelation

- Tests of hypotheses and confidence intervals are unreliable. Regressions may nevertheless provide a good description of the sample. Parameters are unbiased
- Special programs can estimate standard errors consistently
- Include in the model variables affecting neighbouring cases
- Use techniques developed for time series analysis (e.g.: analyse the difference between two points in time, Δy)